ALGEBRAIC THINKING

UNIT 1 COMPONENTS OF ALGEBRAIC THINKING

Introduction

Algebra is the study of patterns and relationships. Finding the value of say, x or y in an equation is only one way to apply algebraic thinking to a specific mathematical problem. The potential for students to think algebraically resides in many of the arithmetic problems they regularly do in school; it requires only a shift in language or a slight extension of a basic arithmetic problem to open up the space of algebraic thinking for students. Algebra is a powerful branch of mathematics as it enables humans to model real-life situations which occur with some regularity and also simplifies patterns in the form of rules.

The NCTM *Principles and Standards for School Mathematics* (2000) includes a description of algebra that goes beyond manipulating symbols. In the *Standards*, algebra is defined as:

- Understanding patterns, relations, and functions;
- Representing and analyzing mathematical situations and structures using algebraic symbols;
- Using mathematical models to represent and understand quantitative relationships and
- Analyzing change in various contexts

Although it may include variables and expressions, algebraic thinking has a broader and different connotation than the term algebra.

What is algebraic thinking?

John Van de Walle, (2004), gives a useful definition of algebraic thinking:
"***Algebraic thinking*** involves representing, generalizing, and formalizing patterns and regularity in all aspects of mathematics." (p. 417). Algebraic thinking may also be defined as, "the use of any of a variety of representations that handle quantitative situations in a relational way. It is "the ability to operate on an unknown quantity as if the quantity was known, in contrast to arithmetic thinking which involves operations on known quantities. The algebraic thinking could be considered to be the "capacity to represent quantitative situations so that relations among variables become apparent" (Steele & Johanning, 2004: 65). Algebraic thinking consists of more than just learning how to solve for the variables x and y; it helps students think about mathematics at an abstract level, and provides them with a way to reason about real-life problems. Algebraic thinking also has problem solving as a point of reference for thinking about algebra and how problem solvers model problems.

Developing Algebraic Habit of Mind

Algebra comprises many mathematical features. Algebraic thinking does not limit to "the ability to operate on an unknown quantity as if the quantity was known..." When people think algebraically certain habits of thinking come into play. The ability to think about functions and how they work, and to think about the impact that a system's structure has on calculations are facilitated by three habits of mind identified to be critical to developing power in algebraic thinking:

- Doing-Undoing,
- Building Rules to represent Functions, and
- Abstracting from computation

1) Doing-Undoing

Reversibility plays a key role in effective algebraic thinking. Encourage students to undo mathematical processes as well as do them (backtracking). Encourage students to work backwards from the answer to the starting point. This helps them to deeply understand the problem. Once a student can solve the equation $2x^2 - 8 = 0$ he/she should be able to answer the question "Write an equation that has solutions -2 and 2 ".

Be guided by the following questions:

- Which process reverses the one I am using?
- What if I start at the end?
- How is this number in the sequence related to the one that came before?

2) Building Rules to represent Functions

This has to do with recognizing patterns and organizing data to represent situations in which input is related to output by well-defined functional rules. For instance, the think of a number game such as "Take an input number, multiply it by 4 and then subtract 3." This is naturally a complement of doing-undoing habit of mind. The capacity to understand how a functional rule works in reverse generally makes it more accessible and useful process.

Guiding questions:

- Is there a rule or relationship here?
- How does the rule work and how is it helpful?
- How are things changing?
- Can I write down a mathematical rule?
- When I do the same thing with different numbers, what still holds true? What changes?

- How does the numbers in the equation relate to the problem context?

3) Abstracting from computation

This has to do with thinking about computations independently of particular numbers that are used. One evident characteristic of algebra is its abstractness, a lot of abstractions take place in algebra. Thinking algebraically involves being able to think about computations freed from the particular numbers they are tied to in arithmetic, i.e., abstracting system regularities from computation. This habit of mind comes into play when students are able to realise for example that they can regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute: 1 + 2 + 3 + … + 100." (Refer to **Gauss'** approach). Recognising that 101 can be decomposed into 100 +1; 99 + 2; 98 + 3; and so on.

Some guiding questions:

- How is this calculating situation like/unlike that one?
- How can I predict what is going to happen without doing all the calculations?
- When I do the same thing with different numbers, what still holds true? What changes?
- How does this expression behave like the other one?

The role of Classroom Questions

In developing habits of mind the thinker should be guided to pay attention, over and over, to "what works" and "how is it working" by looking for cues in new situations. In such cases, previously used strategies may help. This is attained by the individual repeatedly asking three basic algebraic-thinking guiding questions like:

i) How does this process work in reverse?
ii) How are things changing in this situation?
iii) What are my operation shortcut options to get from here to there?

These questions may spur the representation of functions (Questions (i) and (ii)), and abstract thinking about calculation (Question (iii)).

The role of classroom questions by the teacher during instruction is very paramount in developing algebraic thinking.

1) Teacher should consistently model algebraic thinking by making explicit what students might have left implicit in their thinking whenever the teacher is summarizing student responses to a mathematical activity.
2) Teacher should give well-timed pointers to students to help them shift or expand their thinking. That is, teacher should give hints/suggestions for extension at appropriate times. This enables students to pay attention to what is important.

For example, "Once you have made a chart/table, look for an easier way, check how the numbers group and how the grouping might suggest an easier way".

3) Teacher should make it a habit to ask a variety of relevant questions to help students to organize their thinking. Pose questions that challenge students to analyse expressions. E.g., "Can you explain what the 5 and 3 represent in the equation $y = 3x + 5$?"

For each question that the teacher asks, there is the need to make the (i) intention and (ii) the mathematical context clear to both the teacher and students. Make sure that the intentions of the questions asked should be balanced and the questions are asked in situations that are patently algebraic.
It is likely that any algebraic potential will go unexploited unless the teacher asks questions that are used to extend students' thinking about the problem. The teacher thus has to:

1) Reverse a routine calculating task to challenge students to undo as well as do: "Now that you can handle the factor tree, what whole numbers have three factors, including 1?"
2) Ask "what if" questions to extend beyond a single situation to a more generalized situation.

Activity 1: Three Guards in an orchard:

Three guards were protecting an orchard. A thief met the guards, one after the other. To each guard he gave half the apples he had at the time and two extra. Eventually he escaped with just **one** apple. How many apples did the thief originally take?

Follow up questions for extension:

What if he had 2 left? 3 left? 4 left? ……….

3) Exploit calculating situations in which there is a regularity, to challenge students to use calculating shortcuts based on the regularity. E.g.,

   "Without writing out all the numbers and adding them, find the total:

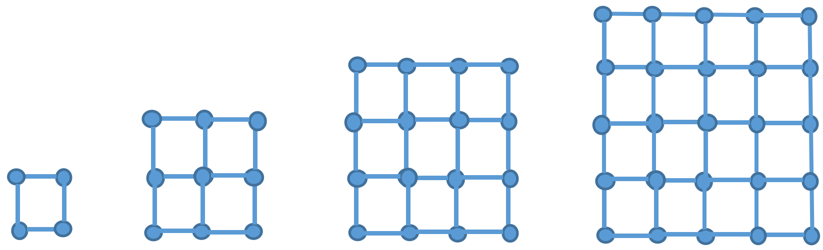
   $$1 + 2 + 3 + 4 + \ldots + 26 + 27 + 28 + 27 + 26 + \ldots + 4 + 3 + 2 + 1."?$$

4) Exploit calculating situations in which there is regularity, to challenge students to make general statements. (e.g., Think of three consecutive integers and multiply them. Does 2 divide any such product? Why? What other integers divide any such product? What is the largest integer that you can be certain divides any such product evenly? Why?)

Activity 2

Toothpick squares (Growing squares made from toothpicks)

1) Study the pattern and draw a picture of the next likely shape in the pattern



2) How many small squares make up the new square?
3) How many small squares would make up a large square which has 10 toothpicks on each side? Show your work.
4) Write a rule which will allow you to find the number of small squares in any large square.
5) Find a rule which will let you find the number of toothpicks in any large square. Show your work.

Notice that you can explore three components of algebraic thinking: (1) making generalizations, (2) conceptions about the equals sign (equality), and (3) thinking about unknown quantities. These three components of algebraic thinking provide a useful framework for recognizing whether students are thinking algebraically, and for determining whether a problem can be viewed algebraically.

Generalisation

Prominent in most definitions of algebra is the notion of "patterns." The ability to discover and replicate mathematical patterns is important throughout mathematics. We need to make our students/learners investigate numerical and geometric patterns and express them mathematically in words or symbols. They should analyze the structure of the pattern and how it grows or changes, organise and analyse this information systematically, and to develop generalisations about the mathematical relationships in the pattern. Young students can have meaningful experiences with generalizing about patterns, even though they may not usually express their mathematical ideas using variables and standard functions. For example, when exploring a pattern such as 1, 3, 5, 7, 9,…, young students may make the following observations:

(i) "If you add 1 to an even number, you always get an odd number"
(ii) "If you add 2 to an odd number, you always get another odd number"
(iii) "If you start at 1 and keep adding 2, you get all the odd numbers"
(iv) "If you can separate a number into two equal groups, it's an even number. If one's left over, it's an odd number."

All of these observations are ways of thinking about a simple pattern—the progression of positive odd integers. However, they also provide evidence of algebraic thinking, because each description relies on some sort of generalization that can be applied to any number. Notice that the student is generalizing that no matter how large or small the even number, adding one (1) will create an odd number as in (i). In observation (iv), the student has identified the property that any even number can be split into equal groups, but odd numbers cannot. Both of these observations are examples of generalization, since they are **projecting a mathematical property onto a whole category of numbers**; in this case, "the even numbers." It may take some time for students to develop strategies for justifying a pattern.

The **first step is noticing that there is a pattern** in a number sequence, and then wondering if that pattern continues as the numbers get larger.
The next step is to describe the pattern, **followed by extending it**. Eventually, students will arrive at a generalized understanding of the pattern; they will be able to predict whether a specific number (or term) is part of a pattern without calculating each consecutive term. For example, given the pattern 1, 3, 5, 7, 9, …, above, students will be able to determine that a number such as 381 is part of the pattern because it is an odd number, and will not need to write out each odd number from 1 to 381 to be convinced of this fact. Later, most students will be ready to work on proving statements such as "adding 2 to an odd number produces another odd number," but their ideas about proof will continue to evolve.

From a formal algebraic perspective, all four statements above follow from the fact that all odd numbers are of the form $(2n + 1)$ where n is a whole number, but students can make and test conjectures long before they ever see such an expression. It is important to keep in mind that as students propose generalizations such as those above, they may be basing their claims on only one or two instances of a pattern. Mathematically this is not enough evidence to determine whether a pattern exists. In observation 3 above, for example, a student may have noticed that 1 + 2 = 3 (adding 2 to an odd number produced another odd number), and 3 + 2 = 5, also an odd number. However, she may not have investigated any numbers beyond those. It is important for elementary students to learn that forming generalizations from only a few instances can lead to inaccurate conclusions.

One example of this can be seen in students' solutions to the "Spider climbing a wall" activity. It reads, *A spider is trying to climb a wall that is 15 metres high. In each hour, it climbs up 3 metres, but falls back 2 metres. In how many hours will it reach the top of the wall?*
Explain your answer.

The students try to use generalization to solve this problem, and figure that the spider climbs 2 metres total for each 2-hour period because he climbs up 3 metres in the first hour and slips down 1 metre in the second hour. Using this generalization, they come to the conclusion that it will take the spider 15 hours to reach the top of a 15-metre wall. However, while the relationship holds in general for each 2-hour period, the 14^{th} hour occurs in the middle of a 2- hour period. During this hour the spider reaches the top of the wall and climbs out, and consequently does not "slide down."

While students have made a generalization that is true in most cases, they have neglected to notice that their current problem is an exception to the general rule of up three, down two. In the end, their understanding of the relationship actually misleads them into solving the problem incorrectly.

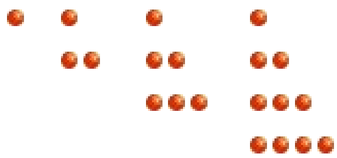
Generalising that $\mathbf{1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}}$

Showing that the **n-th** Triangular number, $1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$.

The first approach is a **visual** one involving only the formula for the area of a rectangle. This is followed by two proofs using algebra.

(i) A visual proof that $\mathbf{1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}}$

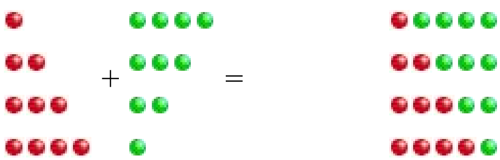
We can visualize the sum $1 + 2 + 3 + 4 + \ldots + n$ as a **triangle of dots**. Numbers which have such a pattern of dots are called **Triangle (or triangular) numbers**, written T(n), the sum of the integers from 1 to n :

N	1	2	3	4	5	6
T(n) as a sum	1	1+2	1+2+3	1+2+3+4	1..5	1..6
T(n) as a triangle					...	
T(n)=	**1**	**3**	**6**	**10**	**15**	**21**

For the proof, we will *count the number of dots in T(n)* but, instead of *summing the numbers 1, 2, 3, etc up to n* we will find the total using only *one multiplication and one division!*

To do this, we will fit **two copies of a triangle of dots together**, one red and an upside-down copy in green.

E.g. T(4)=1+2+3+4

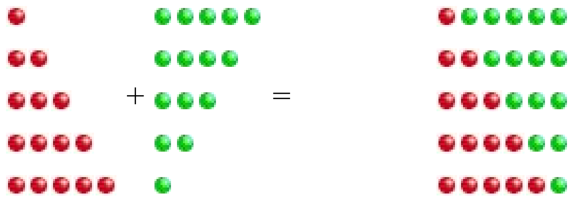


Notice that

- we get a **rectangle** which has the same number of rows (4) but has one extra column (5)
- so the rectangle is **4 by 5**
- it therefore contains 4x5=20 balls
- but we took **two** copies of T(4) to get this
- so we must have 20/2 = 10 balls in T(4), which we can easily check.

This visual proof applies to any size of triangle number.

Check for T(5):



So T(5) is half of a rectangle of dots 5 tall and 6 wide, i.e. half of 30 dots, so T(5)=15.

We might write out the above proof using algebra:
T(n) = 1 + 2 + 3 + 4 + 5 + … + n

T(n)+T(n)= 1 + 2 + 3 + … + (n-1) + n adding for the first T(n)
+ n + (n-1) +(n-2) +…. +2 +1 adding in reverse order for the second T(n)

2T(n) = (n+1) + (n+1)+ (n+1)+ (n+1)+ (n+1)++…….+ (n+1) that is, (n+1) n times

2T(n) = n(n+1)………thus there are 'n' terms of (n+1)

$T(n) = \frac{n(n+1)}{2}$………………. Dividing through by 2

Hence $Tn = \frac{1}{2}n(n+1)$ as the generalization for triangular numbers

(ii) Using the Sigma notation
To be more precise we can use the **sigma notation** in summing a series.
Mathematicians use the capital sigma for the **sum** of a series as follows:

$$\sum_{i=\text{starting value}}^{i=\text{final value}} i$$

- The formula describes the i^{th} term of the series being summed. It is written *after* the sigma;
- The starting value for i is written *below* the sigma;
- The ending value for i is written *above* the sigma

Often the variable is omitted *above* the sigma but ***never omitted*** below the sigma.
Here are some examples:

- The sum $10^2+11^2+12^2$, where the numbers added are the **square numbers** i^2:

$$\sum_{i=10}^{i=12} i^2$$

- The same sum can also be written in many other ways, for instance, as the sum of the **square numbers** $(i+9)^2$ where this time i goes from 1 to 3

$$\sum_{i=1}^{i=3} (i+9)^2$$

The sum 1+2+3+..+9 is T(9) is given by

$$T_9 = \sum_{i=1}^{i=9} i$$

Here is T(n) which is 1+2+3+...+n, this time omitting the second use of the i above the sigma:

$$T_n = \sum_{i=1}^{i=n} i$$

T(n) can also be written backwards:
n+ (n-1) + ... 3 + 2 + 1 where the i^{th} term is now n+1–i for i from 1 to n:

$$\sum_{i=1}^{i=n} (n+1-i) = T_n$$

Note that if all the terms are *independent of the variable*, that is, if there is no *i* in the formula but the variable below the sigma is *i*, then all the terms are ***constant***.
The *number of terms* will be given by the starting and ending values. Here, all the terms are fixed (constant) at 3:

$$\sum_{i=4}^{i=7} 3 = 3+3+3+3 = 12$$

Equality

The arithmetic problem " 5 + 24" could just as well be stated "5 + 24 = ?" or "5 + 24 = ❑" or even "5 + 24 = *x*." These notations create a connection between arithmetic and the "missing value" image of algebra. Consider the algebraic statement "5 + 24 = ? + 15." On the face of it, this expression is similar to the previous ones (e.g. 5 + 24

= ?) but there is one very important difference: the number that replaces the ? is no longer 29, but a smaller number that when added to 15, produces 29.

The issue resides in the meaning students assign to the "=" sign. In the case of the problem "5 + 24 = ?" the "=" can be thought of as "the result of the computation" or a "**do something**" "5 + 24 = ?"." Another interpretation of the equal to sign arises in an example such as, 5 + ? = 24. In this case, the notion of the equal to sign as 'balancing' is important because it calls for determining the value that has to be added to 5 in order to give the result 24? However, in the example "5 + 24 = ? + 15," the equals sign must be interpreted differently. It is now a statement of **equivalence** between two quantities, in this case between "5 + 24" and "? + 15."
Now the ? must be replaced by something other than 29, since "5 + 24" and "29 + 15" are not equivalent. *Understanding that the sign "=" requires that one expression be **equivalent** to the other is a basic tenet of algebra.* Our students should be made to see a variety of problems with unknowns in different positions, such as:

- 4 + ? = 17 + 2
- ? + 15 = 12 + 32
- 13 + 24 = 50 + ?
- 13+ ? = 42

Unknown quantity

The words **"variable,"** and **"unknown"** are words most frequently associated with algebra. Along with this concept comes the idea that the "unknown" will eventually become "known". This is what solving equations is usually about. But it is possible (and important) for students to work with expressions that include a variable that remains unknown. Most number tricks of the form, "choose a number, multiply it by 3, add 6, divide by 3, subtract 2 and tell me the number – and I'll tell you your original number," can be expressed algebraically without the need to use a specific number. The algebraic component is that the trick works for all numbers, not just a specific one for which we have to solve.

Here's an example of a problem with an unknown quantity that remains unknown.
Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. Abena's mother gives her 5 more pieces of erasers. Now who has more? How many more? Then Abena gives Ama one of her pieces of erasers. Now who has more? How many more?
Students can solve this problem without creating algebraic expressions that contain variables. They may draw a picture to represent the number of erasers Abena has (e.g. a circle), and then represent Ama's erasers with a circle and three extra X's. They could then manipulate the pictures without ever specifying what is in the circle.
In this problem, finding the exact amount of erasers Abena has is not important, since the
problem asks for a comparison between two quantities.

Abena Ama XXX	Abena XXXXX Ama XXX
In this diagram, the amount of erasers that Abena and Ama begin with is represented as ovals. The extra pieces that Ama has are represented as x's. This diagram shows that Ama has more erasers, since he has 3 x's, and Abena has none.	In this diagram, Abena has been given 5 more pieces of erasers, which are represented by x's. Since she has more x's (or individual pieces of erasers) than Ama, she must also have more total erasers, because the quantities in the ovals are the same.

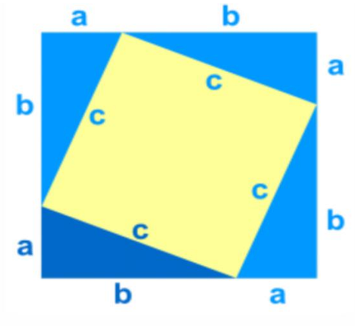
However, some problems similar to the one above cannot be solved without figuring out the value of an unknown number of erasers. For example: *Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. If Abena gets more erasers so that she has twice as many as before, who has more erasers now? How many more?*
There isn't a single answer to this problem; it depends on how many erasers Abena had to begin with. So, if Abena had 2 erasers originally, Abena will now have 4, and Ama will have 5. On the other hand, if Abena begins with 5 erasers, then she will now have 10, while Ama has 8.

The difference between these two kinds of problems is subtle, but as students approach higher grades, they should be able to start making the distinction and solving them appropriately. These types of problems help develop algebraic thinking skills because they require students to think flexibly about quantities, and to learn how to compare related quantities. They also promote the idea that the relationship between two quantities (here, whether Abena or Ama has more erasers) can change depending on how the original amount is acted upon.
As students encounter more complex linear equations, they will be able to interpret the *"=" sign as an indication of equality, not as a sign requiring them to always compute something.* They will have already considered the kinds of patterns that they may now be asked to express in algebraic form. And they will be prepared to work flexibly with variables as unknown quantities rather than needing to figure out its value immediately. With these insights in hand, students will find that algebra is not a mystery, but a territory that already has familiar landmarks.

Proof of the Pythagorean Theorem using Algebra

We can show that $\mathbf{a^2 + b^2 = c^2}$ using Algebra.

Take a look at the following diagram. It has four triangles (right-angled) with sides ‘a’ ‘b’ and ‘c’ enclosing a small tilted square of side length ‘c’. The big square is of length (a + b).



Area of Whole Square

Now area of the big square is $A = (a + b)(a + b)$

Area of the Pieces

Let's add up the areas of all the smaller pieces:

First, the smaller (tilted) square has an area of: c^2

Each of the four triangles has an area of: $\frac{1}{2}ab$

So all four of them together is: $4 \times \frac{1}{2}ab = 2ab$

Adding up the tilted square and the 4 triangles gives: $A = c^2 + 2ab$

But both areas must be equal.

The area of the **large square** is equal to the area of the **inner tilted square** and **the 4 triangles**.

This can be written as:

$$(a + b)(a + b) = c^2 + 2ab$$

Let us now rearrange this to see if we can get the Pythagoras theorem:

Start with $(a + b)(a + b) = c^2 + 2ab$

Expand $(a + b)(a + b)$ to get $a^2 + 2ab + b^2 = c^2 + 2ab$

Subtract "$2ab$ " from both sides: $a^2 + b^2 = c^2$

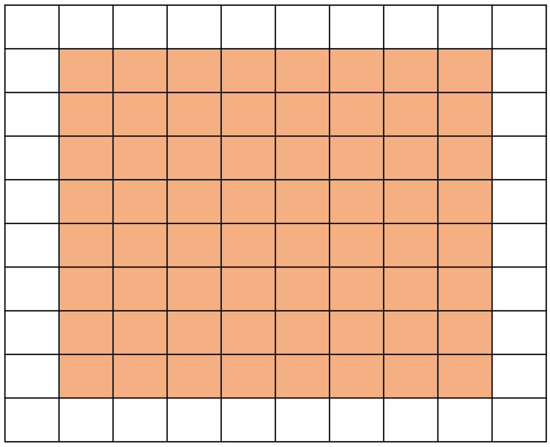
Self-Assessment Questions

1. Explain the three habits of the mind useful in algebraic thinking.
2. Distinguish between the terms ‘unknown’ and ‘variable’ as used in mathematics education.
3. Explain how you lead your students to solve the following problem showing clearly the algebraic processes involved:

a) **The Border Problem**

(i) Show students the following 10 x 10 square grid for a few seconds. Without counting, ask them to determine how squares are in the border of the 10 x 10 grid.

Use **at least three** different approaches and explain your reasoning for each approach.



(ii) How many squares will there be in the border of any square grid? How do you know your rule works?

b) **The River crossing Problem**

A farmer must transport a fox, goose and bag of beans from one side of a river to another using a boat which can only hold one item in addition to the farmer, subject to the constraints that the fox cannot be left alone with the goose, and the goose cannot be left alone with the beans. What is the fewest number of trips the farmer has to take in order to transport all three safely to the other side of the river?
Show working using any appropriate mathematical representation(s)

c) **The Boat Problem**

Eight adults and two children need to cross a river. A small boat is available that can hold one adult or one or two children. Each can row the boat.

(i) How many one-way trips does it take for all of them to cross the river?
(ii) What if there are 2 children and 100 adults?
(iii) What if there are 2 children and any number of adults?
(iv) What happens if there are different numbers of children? For example, 8 adults and 2 children? 8 adults and 4 children?
(v) Write a rule for finding the number of trips needed for X adults and Y children.
(vi) Explain why the rule works in at least two different ways.

d) **The Postage Problem**

The post office only sells stamps of denominations Gh¢3 and Gh¢5. They have unlimited supply of both types of stamps. However, they will only mail your letter or package if the amount of postage you need can exactly be paid with the stamps they have available. For example, you cannot mail Gh¢1 letter or Gh¢4 letter.

(i) What amounts of postage can you buy?
(ii) What amounts of postage are not possible?
(iii) What amounts of postage can you buy if the denominations are Gh¢8 and Gh¢11?
(iv) What generalisations can you make for stamp denominations m Gh¢ and n Gh¢ where m and n are positive integers? Justify your answer.